THOMSON SCATTERING TAKEN INTO ACCOUNT IN THE RELATIVISTIC TRANSFER EQUATIONS FOR A GREY-BODY AND STATIONARY SHOCK WAVE STRUCTURE

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Exact relativistic transfer equations for components of the energy-momentum tensor of the radiation are obtained on the basis of the relativistically covariant radiation transfer equation. Here the absorption and scattering coefficients of the radiation by the medium, which is taken to be a real gas, are considered to be independent of the frequency of the radiation. Eddington's assumption is used as the angular approximation. The system of equations thus obtained is applied in order to investigate the structure of a stationary shock wave of amplitude greater than the critical. A qualitative picture is obtained of the variation of hydrodynamic and radiation characteristics over the entire shock wave zone. It is found that in the case when scattering predominates over absorption the radiation acts on the gas like a non-transparent piston and in doing so limits the radiation damping of the shock wave.

Considerable velocities in the macroscopic motion of high temperature gas and large radiation energy densities are often characteristic in astrophysical phenomena. The relativistically covariant transfer equation which takes these effects into account in the general case was first obtained by Thomas [1]. In the case of one dimensional motion in a fixed system of coordinates it has the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}\right)I_\nu = -\alpha_{\nu_0}I_\nu L+\alpha_{\nu_0}\frac{B_{\nu_0}}{L^2}-\sigma_{\nu_0}I_\nu L+$$
$$+\frac{1}{2}\int_{-1}^{1}d\mu_1\int_0^\infty\sigma_{\nu_1}\Omega(\mu,\nu,\mu_1,\nu_1)\,I_{\nu_1}(\mu_1)\,d\nu_1,$$
$$L=\theta(1-\beta\mu),\qquad \theta=\frac{1}{\sqrt{1-\beta^2}}. \tag{1}$$

Here I_ν is the spectral intensity of radiation of frequency ν; μ is the cosine of the angle between the direction of motion of the gas and that of emission of the radiation; α_{ν_0}, σ_{ν_0} are linear coefficients of radiation absorption and scattering for a gas at rest; B_{ν_0} is Planck's function which appears as the result of local thermodynamic equilibrium being assumed; $\Omega(\mu, \nu, \mu_1, \nu_1)$ is the re-emission function; β is the ratio of the gas velocity to the velocity of light. Quantities in the fixed and particular systems of coordinates are connected by the relationships [1, 2]

$$\nu=\frac{\nu_0}{L},\quad d\nu=\frac{d\nu_0}{L},\quad d\mu=L^2\,d\mu_0,$$
$$\alpha_\nu=\alpha_{\nu_0}L,\quad \sigma_\nu=\sigma_{\nu_0}L,$$
$$\Omega(\mu,\nu,\mu_1,\nu_1)=\Omega_0(\mu_0,\nu_0,\mu_{10},\nu_{10})\frac{L_1}{L^2}$$
$$L_1=\theta(1-\beta\mu_1). \tag{2}$$

We shall now assume that the coefficients of radiation absorption and scattering for a stationary gas are independent of the frequency, i.e., $\alpha_{\nu_0}=\alpha_0$ and $\sigma_{\nu_0}=\sigma_0$ (the approximation of grey-body with Thomson scattering), α_0 and σ_0 are taken to be given functions of gas density and temperature determined by particular conditions. We shall further assume that the scattering is coherent, i.e., it occurs without a change of frequency, and is isotropic. The re-radiation function in the particular system of coordinates may then be written in the form

$$\Omega=(\mu_0,\nu_0,\mu_{10},\nu_{10})=\delta(\nu_0-\nu_{10}).$$

In Eq. (1) we shall transform the re-radiation term in accordance with the rules (2). Since $\nu_{10}=\nu_1 L_1$ and $\nu_0=\nu L$, then

$$\delta(\nu_{10}-\nu_0)=\frac{1}{L_1}\delta\left(\nu_1-\nu\frac{L}{L_1}\right)$$

and consequently we have in the fixed system of coordinates

$$\sigma_{\nu_1}\Omega(\mu,\nu,\mu_1,\nu_1)=\frac{\sigma_0L_1}{L^2}\,\delta\left(\nu_1-\nu\frac{L}{L_1}\right). \tag{3}$$

Using (3) in Eq. (1), we obtain the transfer equation for the spectral intensity of the radiation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}\right)I_\nu=-(\alpha_0+\sigma_0)\,I_\nu L+$$
$$+\alpha_0\frac{B_0}{L^2}+\frac{\sigma_0}{L^2}\frac{1}{2}\int_{-1}^{1}L_1I_{\nu L/L_1}(\mu_1)\,d\mu_1. \tag{4}$$

Integrating (4) with respect to frequency ν and performing the substitution $\nu L/L_1=\nu'$ in the last term, we obtain the transfer equation for the integral intensity of radiation I in the form

$$\left(\frac{\partial}{\partial\tau}+\mu\frac{\partial}{\partial\xi}\right)I=-IL+\alpha\frac{B_0}{L^3}+$$
$$+\frac{\varkappa\theta^2}{L^3}\int_{-1}^{1}I(\mu_1)\,[1-2\beta\mu_1+\beta^2\mu_1^2]\,d\mu_1,$$
$$B_0=\frac{\sigma}{\pi}T_0^4,\quad \alpha=\frac{\alpha_0}{\alpha_0+\sigma_0},$$
$$\varkappa=\frac{\sigma_0}{\alpha_0+\sigma_0}\qquad(\alpha+\varkappa=1),$$
$$d\tau=c(\alpha_0+\sigma_0)\,dt,\quad d\xi=(\alpha_0+\sigma_0)\,dx,\quad I=\int_0^\infty I_\nu d\nu. \tag{5}$$

Here σ is the Stefan-Boltzmann constant, T_0 is the stationary gas temperature.

From this equation it is not difficult to obtain a system of equations for the angular momenta of the intensity I, components of the energy-momentum tensor of the radiation. In order to do this we follow [3] and introduce the symbols:

$$J = \frac{4\pi}{\rho_0 D^2 c}\int_{-1}^{1} I\,d\mu,\ S = \frac{4\pi}{\rho_0 D^2 c}\int_{-1}^{1} I\mu\,d\mu,\ K = \frac{4\pi}{\rho_0 D^2 c}\int_{-1}^{1} I\mu^2\,d\mu,$$

$$T = \frac{T_0}{T_\infty},\quad \delta_1 = \frac{8\sigma T_\infty^4}{3c\rho_0 D^2}. \tag{6}$$

Here ρ_0, D, T_∞ are certain constants having dimensions of density, velocity, and temperature respectively, J and K are the dimensionless radiation energy density and radiation pressure (diagonalized tensor of radiation momentum density flux) in units $\rho_0 D^2$, and S is the dimensionless density of the radiation energy flux in units $\rho_0 D^2 c$.

We shall apply the integral operators

$$\frac{4\pi}{\rho_0 D^2 c}\int_{-1}^{1}(\ldots)\,d\mu,\qquad \frac{4\pi}{\rho_0 D c}\int_{-1}^{1}(\ldots)\,\mu\,d\mu$$

to Eq. (5).

Using the integrals

$$\frac{1}{2}\int_{-1}^{1}\frac{d\mu}{L^3} = \frac{1}{2\theta^3}\int_{-1}^{1}\frac{d\mu}{(1-\beta\mu)^3} = \theta,\qquad \frac{1}{2}\int_{-1}^{1}\frac{\mu\,d\mu}{L^3} = \beta\theta$$

we obtain

$$\frac{\partial J}{\partial\tau} + \frac{\partial S}{\partial\xi} = \theta^3\{\beta[\beta(J+K) - S(1+\beta^2)] + \alpha[B_0 - J + 2\beta S - \beta^2(B_0+K)]\},$$

$$\frac{\partial S}{\partial\tau} + \frac{\partial K}{\partial\xi} = \theta^3\{[\beta(J+K) - S(1+\beta^2)] + \alpha\beta[B_0 - J + 2\beta S - \beta^2(B_0+K)]\}. \tag{7}$$

The right sides of these equations become physically clear if we use the tensor relations between components of the energy-momentum tensor in various coordinate systems [2]:

$$J_0 = \theta^2(J - 2\beta S + \beta^2 K),$$
$$S_0 = \theta^2[S(1+\beta^2) - \beta(J+K)],$$
$$K_0 = \theta^2(K - 2\beta S + \beta^2 J). \tag{8}$$

Then Eqs. (7) assume the form

$$\frac{\partial J}{\partial\tau} + \frac{\partial S}{\partial\xi} = \theta[-\beta S_0 + \alpha(B_0 - J_0)],$$

$$\frac{\partial S}{\partial\tau} + \frac{\partial K}{\partial\xi} = \theta[-S_0 + \alpha\beta(B_0 - J_0)]. \tag{9}$$

If we neglect the relativistic dependence of temperature, we may write $B_0 = 3\delta_1 T^4$. If the temperature T_∞ is understood to be the temperature for which there is equilibrium between the radiation and the gas, then the equilibrium condition in the characteristic coordinate system is $S_0 = 0$; $J_0 = 3\delta_1$, $K_0 = \delta_1$, and the corresponding equilibrium conditions in the fixed coordinate system is according to (8)

$$J = \frac{3\delta_1}{1-\beta^2}\left(1+\frac{\beta^2}{3}\right),\quad S = \frac{4\beta\delta_1}{1-\beta^2},\quad K = \delta_1\frac{1+3\beta^2}{1-\beta^2}. \tag{10}$$

System (9) is not closed. In order to close it we must postulate an additional relation between the radiation quantities. For a gas which is at rest this relation is taken to be $J_0 = 3K_0$, the familiar Eddington approximation. This is an extension to non-equilibrium conditions of a relation which is strictly valid only in equilibrium conditions. We may take the relation

$$J = K\frac{3+\beta^2}{1+3\beta^2}, \tag{11}$$

which comes from the equilibrium values (10), as a similar relation in the fixed coordinate system.

Thus Eqs. (7) and (11) are the relativistic equations for radiation in Eddington's modified approximation. In the non-relativistic form these equations are

$$3\frac{\partial K}{\partial\tau} + \frac{\partial S}{\partial\xi} = \alpha[3(\delta_1 T^4 - K) - 2\beta(4\beta K - S)] + \beta(4\beta K - S),$$

$$\frac{\partial S}{\partial\tau} + \frac{\partial K}{\partial\xi} = 3\alpha\beta(\delta_1 T^4 - K) + (4\beta K - S) \tag{12}$$

with the boundary conditions

$$K = \delta_1,\qquad S = 4\beta\delta_1. \tag{13}$$

Terms of the order β^2 are kept in the first of these equations in order that the given boundary conditions (13) should be satisfied exactly.

We shall apply the system of equations which we have obtained to the problem of the structure of the front of a strong stationary shock wave. Let the medium in which the shock wave is moving with a constant velocity D be an infinite layer of ideal gas of density ρ_0, with an adiabatic index $1 < \gamma < 2$ (to allow for the effective ionization of the matter), and with a zero initial temperature. After the passage of the shock wave the gas has a final temperature T_∞ and density ρ_1. Thus the quantities entering into the definition (6) are given a clear physical meaning. The initial parameters of the gas correspond to the optical coordinate $\xi = +\infty$, the final parameters to the coordinate $\xi = -\infty$. The discontinuity is situated in the center of the coordinate system $\xi = 0$.

We introduce the definitions

$$\eta = \frac{\rho_0}{\rho},\quad A = \frac{RT_1}{\mu D^2},\quad r = \frac{\gamma-1}{\gamma+1},$$

$$q = \frac{D}{c},\quad G = r\left(\frac{S}{q}+1\right), \tag{14}$$

where ρ is the density of the gas, R is the universal gas constant, μ is the molecular weight of the gas. Then in the accepted notation the first integrals of radiation hydrodynamics have the form [3]

$$\beta = -q\eta,\qquad AT = \eta\left(1 - \eta - \frac{1}{2}K\right),$$

$$G = AT\frac{1+r}{r} + \eta^2. \tag{15}$$

Using these relations the system of stationary radiation Eqs. (12) assumes the form

$$\frac{dG}{d\xi} = -\frac{r}{q}\left[3\alpha(K - \delta_1 T^4) + q^2\eta(2\alpha - 1)\left(\frac{G}{r} - 1 + 4K\eta\right)\right]$$

$$\frac{dK}{d\xi} = q\left[3\alpha\eta(K - \delta_1 T^4) - \left(\frac{G}{r} - 1 + 4K\eta\right)\right]. \tag{16}$$

The boundary conditions for system (16) have the form

$$K = T = 0, \quad \eta = 1, \quad G = r, \quad \xi = +\infty,$$
$$K = \delta_1, \quad T = 1, \quad \eta = \eta_1, \quad G = r(1 - 4\delta_1\eta_1),$$
$$\xi = -\infty. \tag{17}$$

The properties of system (16) for the case $\alpha = 1$ (pure absorption without scattering) have been investigated in detail in a paper by Morozov.*

It is peculiar to this case that in the phase planes (G, K) or (T, η) the treatment gives us the absorption coefficient α_0 as a concrete function of the gas parameters. Actually the nonlinear equation to be investigated in the phase plane (G, K) has the form

$$\frac{dG}{dK} = -\frac{r}{q^2}\frac{3\alpha P(\eta) + q^2\eta(2\alpha - 1)Q(\eta)}{3\alpha\eta P(\eta) - Q(\eta)},$$
$$P(\eta) = K - \delta_1 T^4, \qquad Q(\eta) = \frac{G}{r} - 1 + 4K\eta. \tag{18}$$

In the case $\alpha = 1$ the properties of the absorbing medium enter only into the definition of the optical thickness ξ. It is convenient to carry out the investigation in the phase space (T, η). Using Eqs. (15) we arrive at the relation

$$\frac{dT}{d\eta} = \frac{1}{A}\left[\frac{dG}{dK}\left(1 - 2\eta - \frac{1}{2}K\right) - r\eta^2\right]\left[\frac{dG}{dK} + \eta\frac{1+r}{2}\right]^{-1}. \tag{19}$$

Setting (18) in this equation we obtain

$$\frac{dT}{d\eta} = \frac{1}{A}\Big[3\alpha(1 - 2\eta - {}^1\!/_2K)(1 + q^2\eta^2)P(\eta) +$$
$$+ q^2\eta[(2\alpha - 1)(1 - 2\eta - {}^1\!/_2K) - \eta]Q(\eta)\Big]\times$$
$$\times\Big[3\alpha[1 - q^2\eta^2(1+r)/2r]P(\eta) + q^2\eta[2\alpha + (1-r)/2r]Q(\eta)\Big]^{-1}. \tag{20}$$

It is clear from this that if α is equal to or of the order of 1, then for the condition $K \gg \delta_1 T^4$ (state of strong non-equilibrium) we may neglect terms of the order of q^2 and the resulting equation has as its first integral the expression $(AT + \eta^2)/\eta = \text{const}$. According to (15) this is just the condition of constant radiation pressure. Consequently in conditions far removed from equilibrium, the integral curve should be close to the curve of constant K [4]. However, the state of the gas changes between the bounding points for equilibrium determined by conditions (17). Thus the change of radiation pressure K from 0 to δ_1 should occur close to the curve of radiative equilibrium $K = \delta_1 T^4$. In the plane (T, η) this equation has the form

$$T^4 + 4mT - 3n = 0, \qquad m = \frac{A}{2\delta_1\eta}, \qquad n = \frac{2(1-\eta)}{3\delta_1}. \tag{21}$$

The solution of this equation is

$$T = \left(\frac{m}{\sqrt{y}} - y\right)^{1/2} - \sqrt{y},$$
$$y = \frac{1}{2}\left\{\sqrt[3]{\sqrt{m^4 + n^3} + m^2} - \sqrt[3]{\sqrt{m^4 + n^3} - m^2}\right\}. \tag{22}$$

A qualitative investigation of the singular points of (17) shows that for all possible parameters of the problem the initial point remains a saddle point.

If the inequality

$$A > \eta^2{}_1(1-r)(1+r)^{-1}$$

is fulfilled the final equilibrium point is also a saddle. Physically this inequality means that at the final equilibrium point the gas velocity is less than the adiabatic velocity of sound. In this case a discontinuous solution should result. Before the discontinuity the integral curve lies close to the curve of radiative equilibrium (22), practically coinciding with it (to the order of q^2), while behind the shock the integral curve lies close to the curve of constant radiation pressure equal to the limiting value δ_1. It follows from (15) that this curve is given by the equation

$$AT = \eta(1 - \eta - {}^1\!/_2\,\delta_1). \tag{23}$$

It is clear from here that the temperature behind the shock is higher than the temperature at the final point and may attain a maximum value T_m at the point η_m,

$$T_m = \frac{1}{4A}\left(1 - \frac{1}{2}\delta_1\right)^2, \qquad \eta_m = \frac{1}{2}\left(1 - \frac{1}{2}\delta_1\right), \tag{24}$$

if this point lies in the region behind the shock.

The radiation flux in the region after and before the shock is determined respectively by the expressions

$$G = \eta\,[(1 + r)(1 - {}^1\!/_2\,\delta_1) - \eta],$$
$$G = \eta\,[(1 + r)(1 - {}^1\!/_2\,\delta_1 T^4) - \eta]. \tag{25}$$

Here $T(\eta)$ is calculated from (22).

If we turn to the initial equations (16) we see that the optical thickness of the equilibrium zone before the shock is of the order q/δ_1, i.e., this zone is almost transparent and the temperature peak behind the shock is optically very thin.

As mentioned above such a picture of the behavior of the integral curves is obtained for α of the order of unity. However it follows from Eqs. (18) that the picture will remain the same for small values of α of the order q. Here the departure of the integral curves from the curves of radiative equilibrium and constant K will also be of the order of q (more accurately, the order of departure will be equal to q^2/α).

In the case of very small α (i.e., in conditions when there is appreciably more scattering than absorption, which occurs for very high temperatures) the behavior of the integral lines presents quite another picture.

First of all the case $\alpha = 0$ (pure scattering) is rather special since in this case the initial nonlinear equation has no singular points corresponding to points of equilibrium. For $\alpha = 0$ Eq. (20) may be integrated exactly:

$$T = \text{const}\cdot\eta^{-\frac{2r}{1-r}} = C\eta^{1-\gamma}. \tag{26}$$

This solution corresponds to adiabatic variation of the thermodynamic parameters of the gas. Here the radiation becomes independent of the material and acts upon it like a piston. On account of the scattering the radiation momentum is transferred directly to the medium, and thus the scattering is a factor hindering the radiation damping of the shock wave.

In the region where scattering predominates the behavior of the quantities is described by the equations

$$K = 2\left[1 - \eta - C\eta^{-\frac{1+r}{1-r}}\right],$$
$$G = \eta\left[(1 + r)C\eta^{-\frac{1+r}{1-r}} + \eta r\right] \tag{27}$$

and Eq. (26).

*Yu. I. Morozov, Candidate's dissertation: Some Nonlinear Problems of Relativistic Radiation Hydrodynamics, Moscow, 1965.

However, this region cannot include a point of finite equilibrium. Qualitative analysis shows that in a small neighborhood of this point the integral curve lies close to the curve $K = \delta_1$, as in the case $\alpha \approx 1$, and only afterwards at distances of the order of q^2 does it approach the pure scattering curve. If the discontinuous solution is realized, i.e., the inequality

$$A > \eta_1^2 (1 - r)(1 + r)^{-1}$$

is satisfied, then regions of pure scattering cannot be adjacent to both sides of the discontinuity, since the condition for continuity of G and K at the shock give only a continuous solution in this case.

If we examine (20) it is not difficult to obtain an equation for this case which approximately describes the region adjacent to the shock on the side of the unperturbed gas, for the case when scattering predominates ($\alpha \ll q^2$). This region is described by the equation

$$G / r - 1 + 4K\eta = O(\alpha / q^2), \qquad (28)$$

and consequently quantities in this region satisfy the equations

$$AT = \frac{r}{1-7r}(1-\eta)(1-7\eta), \quad K = \frac{2}{\eta}\frac{(1-\eta)(\eta-r)}{1-7r},$$
$$G = r\left[1 - 8\frac{(1-\eta)(\eta-r)}{1-7r}\right]. \qquad (29)$$

It should be noted that in the regions of pure scattering lying on either side of the shock these quantities vary in quite different ways. In fact it follows from Eq. (20) that for these regions

$$\frac{dG}{dK} = -r\eta. \qquad (30)$$

Using Eqs. (15), we may easily obtain

$$\frac{dK}{d\eta} = \frac{2}{\eta^2}\left(AT\frac{1+r}{1-r} - \eta^2\right) = -2\left(1 - \frac{c_s^2}{v^2}\right),$$
$$\frac{dG}{d\eta} = -\frac{2r}{\eta}\left(AT\frac{1+r}{1-r} - \eta^2\right) = 2r\eta\left(1 - \frac{c_s^2}{v^2}\right). \qquad (31)$$

Here c_s is the adiabatic velocity of sound, v is the gas velocity. It follows from the conditions of shock stability that the relation $v^2 > c_s^2$ is valid before the shock (supersonic flow), while behind the shock $v^2 < c_s^2$.

Thus it follows from formulas (31) that in the region of pure scattering the inequalities $dK/d\eta < 0$, $dG/d\eta > 0$ are valid in front of the shock, while those for the region behind the shock are $dK/d\eta > 0$, $dG/d\eta < 0$.

When all the regions mentioned above are matched we obtain an approximate solution for the problem of the shock-wave structure with a sufficiently high degree of accuracy.

In my paper which I refer to on page 29, the structure of a strong shock wave was investigated for the stationary case of propagation through a plane layer of decreasing density. It was shown that in the case of pure absorption radiative effects hinder the building up of shock wave energy, and may not only check an unlimited increase of gas temperature and velocity at the shock as it approaches the boundary of the region, but may even lead to a decrease of these quantities as the boundary is approached.

Our treatment of the effect of radiation scattering on shock wave structure show that it is absolutely essential to take scattering into account in cases of this type, and this may lead to a different qualitative picture since radiative braking of the shock wave will be suppressed. The author is grateful to V. S. Imshennik for his interest and participation in the work.

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